GRASPING FORCE OPTIMIZATION USING DUAL METHODS

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Abstract: One of the basic requirements in grasping and manipulation of objects is the determination of a suitable set of grasping forces such that the external forces and torques applied on the object are balanced and the object remains in equilibrium. This paper presents a new mathematical approach to efficiently obtain the optimal solution of this problem using the dual theorem of non-linear programming. The problem is modeled such that the basic convexity property necessary to apply the dual theorem is satisfied and, then, it is transformed into another one much easier to be solved. Three examples showing the efficiency and accuracy of the proposed methodology are included in the paper.

Keywords: Dexterous manipulation, force distribution problem, optimal grasping forces, mathematical programming.

1. INTRODUCTION

Dexterous manipulation by means of mechanical hands has become a field of great interest in the last two decades. The advantages of these kind of end-effectors over the conventional grippers are, among others, the versatility of grasping objects of different shapes and a better distribution of the grasping forces, avoiding to damage the object.

The determination of the contact points on the object boundary is usually based on the properties of form/force-closure, which provide to the grasp the capability of resisting any external disturbance (Bicchi, 1995). Once the grasp satisfies one of these two properties, the problem is to determine an adequate set of contact forces such that the external forces and torques are balanced and the object remains in equilibrium. This is one of the most basic requirements of a grasp and, usually, it has not an unique solution. Some authors prioritized the simplicity and efficiency of the algorithms versus the optimality of the solution in order to use the algorithms in real-time procedures (Yoshikawa and Nagai, 1991; Saut *et al.*, 2005, among others). Other authors focused their interest in obtaining the optimal solution (for instance, the minimal force necessary to hold the object). The main difficulty of determining the optimal grasping forces is the non-linearity of the friction models, although they can be simplified by using linear approximations. In this case, the Simplex algorithm (Kerr and Roth, 1986) or dual linear programming methods (Cheng and Orin, 1990) can be applied. The accuracy of these methods depends on the number of planes used in the approximation which increases their computational cost. A linear method based on the ray-shooting algorithm that use a large number of planes in

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the approximation with a reasonable computational cost was presented by Liu (1999). The non-linearities of the friction cones were maintained by Nakamura *et al.* (1989) where an off-line algorithm based on the primal form of a non-linear problem was presented. The representation of the frictional constraints as a positive definiteness of a symmetric matrix was proposed by Buss *et al.* (1996) and refined by Helmke *et al.* (2002). Based on this representation, these authors developed gradient flow algorithms and Han *et al.* (2000) developed an algorithm based on Linear Matrix Inequality methods. The algorithms proposed by Buss *et al.* (1996), Helmke *et al.* (2002) and Han *et al.* (2000) require to select an initial solution that satisfies the constraints and the step sizes of the algorithms. A solution to these problems as well as a comparison of the three mentioned algorithms was done by Liu *et al.* (2004). The use of a neural network to solve this problem was proposed by Xia *et al.* (2004).

This paper presents a new mathematical approach to efficiently solve the optimal force distribution problem. The problem is modeled as a non-linear minimization problem such that the objective function is the L_2 norm of the finger forces vector and the constraints are obtained by linearizing the friction cones. This model assures the convexity of the problem implying that the dual theorem of non-linear programming can be applied, and the original problem is transformed into another one much easier to be solved. This method allows to use a large number of planes in the linear approximation without increasing the computational cost of the algorithm, allowing an accurate final solution.

2. PRELIMINARY CONCEPTS

2.1 Problem Statement

Let $\boldsymbol{f}_{c_i} \in \mathbb{R}^3$ be the force exerted by a finger on a contact point $\boldsymbol{p}_i$ of the object boundary with $i = 1, ..., n$ and n being the number of fingers in contact. Consider a local contact frame defined by three orthogonal vectors such that one is the inward normal and the other two are tangent to the object boundary. In this case, $\boldsymbol{f}_{c_i}$ can be expressed with respect to this local contact frame as $\boldsymbol{f}_{c_i} = [f^n_{c_i}\ f^{t_1}_{c_i}\ f^{t_2}_{c_i}]^T$, where $f^n_{c_i}$ is the normal and $f^{t_1}_{c_i}$ and $f^{t_2}_{c_i}$ are the tangent components of $\boldsymbol{f}_{c_i}$ (see Fig. 1). Based on the hard contact friction model, each finger force must satisfy the Coulomb's law in order to avoid finger slippage on the object boundary, i.e.:

$$-\mu(f^n_{c_i})^2 + (f^{t_1}_{c_i})^2 + (f^{t_2}_{c_i})^2 \leq 0 \qquad (1)$$

being μ the friction coefficient. Geometrically, Eq. (1) defines the friction cone where the finger force must lie.

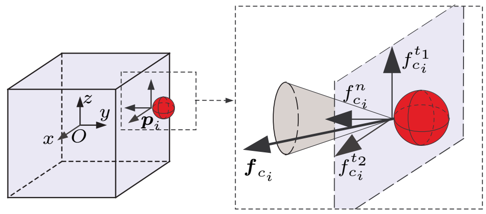

Fig. 1. Object and local contact frames, and hard contact friction model.

All the components of the contact forces generated by the fingers with respect their local contact frames, form the *finger force vector*:

$$\boldsymbol{f}_c = [\boldsymbol{f}^T_{c_1} \cdots \boldsymbol{f}^T_{c_n}] \in \mathbb{R}^{3n} \qquad (2)$$

In order to determine the effect of $\boldsymbol{f}_c$ on the object, it has to be expressed with respect to a common object frame usually chosen at the object center of mass. The relation between the local contact frames and the object frame is defined by the *Grasp Matrix* $\mathcal{G} \in \mathbb{R}^{6\times 3n}$ (Murray *et al.*, 1994). Let $\boldsymbol{w}_{ext} = [f_x\ f_y\ f_z\ \tau_x\ \tau_y\ \tau_z]^T \in \mathbb{R}^6$ be the external wrench vector defined by all the forces and torques exerted on the object with respect to the object frame. In order to maintain the object equilibrium, each finger must exert a contact force such that $-\boldsymbol{w}_{ext} = \mathcal{G}\boldsymbol{f}_c$. The dimension of $\boldsymbol{f}_c$ ($3n$) is greater than the dimension of the external wrench vector (6) when $n \geq 3$, implying that the object is overconstrained and the solution is not unique. In this case, $\boldsymbol{f}_c = \boldsymbol{f}_p + \boldsymbol{f}_h$ with $\boldsymbol{f}_p$ and $\boldsymbol{f}_h$ being two orthogonal vectors given by:

$$\boldsymbol{f}_p = -\mathcal{G}^+\boldsymbol{w}_{ext} \qquad (3)$$

$$\boldsymbol{f}_h = \mathcal{N}\boldsymbol{h} \qquad (4)$$

where $\mathcal{G}^+$ is the pseudo-inverse of $\mathcal{G}$, $\mathcal{N} \in \mathbb{R}^{3n\times(3n-6)}$ is an orthonormal basis of the null space of $\mathcal{G}$ and $\boldsymbol{h} \in \mathbb{R}^{(3n-6)}$ expresses $\boldsymbol{f}_h$ with respect to $\mathcal{N}$. $\boldsymbol{f}_p$ is called the *particular solution* of the problem and this is the required force to balance the external wrench. $\boldsymbol{f}_h$ is called the *homogeneous solution* of the problem or the *internal force vector*. This is the component of $\boldsymbol{f}_c$ that does not contribute to balance the external wrench but it is necessary to maintain the contact forces inside the friction cones. Since $\boldsymbol{f}_p$ is fixed for a set of grasping points and an external wrench, the problem of determining the grasping forces is equivalent to determine $\boldsymbol{h}$.

2.2 Mathematical Programming Background

The main mathematical programming concepts that will be used in this paper to determine the optimal finger forces are summarized in this subsection. The proofs of the conditions and theorems exposed here and further explanations about mathematical programming are given, among others, by Luenberger (1973).

Consider the generic form of a minimization problem with inequality constraints as:

$$\text{Min} \quad \mathcal{F}(\boldsymbol{x}) \tag{5}$$

$$\text{subject to} \quad \boldsymbol{\mathcal{C}}(\boldsymbol{x}) \leq \mathbf{0} \tag{6}$$

where $\boldsymbol{x} \in \mathbb{R}^q$ is the variables vector, $\mathcal{F}$ is the objective function and $\boldsymbol{\mathcal{C}}$ is a p-dimensional function.

Let $\boldsymbol{x^*} \in \Omega$ be a vector that satisfies the constraints defined by Eq. (6). $\boldsymbol{x^*}$ can be:

A *local minimum:* If there is a distance ε such that $\mathcal{F}(\boldsymbol{x}) \geq \mathcal{F}(\boldsymbol{x^*})$, $\forall \boldsymbol{x} \in \Omega$ within $|\boldsymbol{x} - \boldsymbol{x}^*| \leq \varepsilon$.

A *global minimum:* If $\mathcal{F}(\boldsymbol{x}) \geq \mathcal{F}(\boldsymbol{x^*})$, $\forall \boldsymbol{x} \in \Omega$.

A *regular vector:* If the gradients of the constraints $\mathcal{C}(\boldsymbol{x}^*)$ are linearly independent.

While in an optimization problem without constraints the gradient vector of the objective function is null in the local minima, this does not necessarily happen when this function is subjected to constraints. The following conditions generalize the necessary optimality conditions for this kind of problems.

Kuhn-Tucker Conditions (First-Order Necessary Conditions). Let $\boldsymbol{x^*}$ be a local minimum for the minimization problem described by Eq. (5) and (6) and suppose that $\boldsymbol{x^*}$ is regular. There exists a Lagrange multipliers vector $\boldsymbol{\lambda} \in \mathbb{R}^p$ (p being the number of constraints) such that:

$$\nabla \mathcal{F}(\boldsymbol{x}^*) + \boldsymbol{\lambda} \nabla \boldsymbol{\mathcal{C}}(\boldsymbol{x}^*) = \mathbf{0} \tag{7}$$

$$\boldsymbol{\lambda} \boldsymbol{\mathcal{C}}(\boldsymbol{x}^*) = 0 \tag{8}$$

$$\boldsymbol{\lambda} \geq \mathbf{0} \tag{9}$$

where ∇ is the gradient of the respective function. $\diamond$

Let $\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = \mathcal{F}(\boldsymbol{x}) + \boldsymbol{\lambda} \boldsymbol{\mathcal{C}}(\boldsymbol{x})$ be the Lagrangian function of a minimization problem. Eq. (7) is the gradient of the Lagrangian function respect to $\boldsymbol{x}$ and evaluated at $\boldsymbol{x} = \boldsymbol{x^*}$. Eq. (8) and (9) determine the active constraints of the problem, i.e. the inequality constraints that have associated a Lagrange multiplier strictly positive and act just as equalities.

The Kuhn-Tucker conditions are necessary conditions and they have to be always satisfied. In order to determine when the Kuhn-Tucker conditions are also a sufficient condition for a local minimum the following condition is stated.

Second-Order Sufficient Condition. A vector $\boldsymbol{x^*}$ satisfying the Kuhn-Tucker conditions is a strict local minimum of the problem described by Eq. (5) and (6) if the Hessian matrix

$$\mathcal{H}(\boldsymbol{x}^*) = \nabla^2 \mathcal{F}(\boldsymbol{x}^*) + \boldsymbol{\lambda} \nabla^2 \mathcal{C}(\boldsymbol{x}^*) \tag{10}$$

is positive definite on the subspace $M' = \{\boldsymbol{y} : \nabla \mathcal{C}_j(\boldsymbol{x}^*) \boldsymbol{y} = 0 \text{ for all } j \in \boldsymbol{J}\}$, where $\boldsymbol{J} = \{j : \nabla \mathcal{C}_j(\boldsymbol{x}^*) \boldsymbol{y} = 0, \lambda_j > 0\}$ and $\mathcal{C}_j$ is the j component of $\mathcal{C}$. $\diamond$

The methodologies that determine the optimal solution using the original or primal form of the minimization problem are usually called primal methods. Other kind of methods use the dual form of the minimization problem to determine the optimal solution. The dual form transforms the original problem into an equivalent problem where the fundamental unknowns are the Lagrange multipliers and once they are known, the determination of the final solution is simple. The methods based on the dual form are applicable only to a subclass of non-linear optimization problems, since it requires the convexity of the problem. Nevertheless, this property is satisfied in a large range of practical situations and there are important classes of problems for which these methods are better than the primal.

Duality Theorem. Let $\boldsymbol{x^*}$ be a local minimum of the optimization problem described by Eq. (5) and (6), and let $\boldsymbol{\lambda^*}$ be the corresponding Lagrange multipliers vector. Suppose also that $\boldsymbol{x^*}$ is regular and that the Hessian matrix $\mathcal{H}(\boldsymbol{x^*})$ is positive definite. Then, the dual problem

$$\text{Max} \quad \phi(\boldsymbol{\lambda}) = \min[\mathcal{F}(\boldsymbol{x}) + \boldsymbol{\lambda} \mathcal{C}(\boldsymbol{x})] \tag{11}$$

$$\text{subject to} \quad \boldsymbol{\lambda} \geq \mathbf{0} \tag{12}$$

has a local maximum at $\boldsymbol{\lambda^*}$ with corresponding value $\boldsymbol{x^*}$. $\diamond$

Note that the Duality Theorem can be applied only when $\mathcal{H}(\boldsymbol{x^*})$ is positive definite, which assures the convexity of the problem.

3. DETERMINATION OF THE MINIMAL FINGER FORCES

The mathematical programming background presented in Sec. 2.2 is applied here to the problem of determining the minimal grasping forces necessary to balance an external wrench exerted on the object. First, the problem is modeled such that the convexity of the problem is assured. Then, the primal and dual forms of the problem are obtained and, finally, a method to determine the minimal forces based on the dual form is developed.

3.1 Modeling the problem

The minimization of the grasping forces subjected to the friction constraints, can be modeled as a constrained minimization problem described by Eq. (5) and (6). The objective function used in this work is the module of the finger force vector, i.e., $\|\boldsymbol{f}_c\|$, with $\boldsymbol{f}_c$ given by Eq. (2). Taking into account that $\boldsymbol{f}_c$ can be expressed as the sum of two orthogonal vectors ($\boldsymbol{f}_p$ and $\boldsymbol{f}_h$ given by Eq. (3) and (4), respectively) and that $\mathcal{N}$ is an orthonormal basis of the null space of the grasp matrix, $\|\boldsymbol{f}_c\|$ can be expressed as:

$$\|\boldsymbol{f}_c\| = \|\boldsymbol{f}_p\| + \|\boldsymbol{h}\| \tag{13}$$

The friction cones are modeled with the typical linear approximation to a pyramid of m faces. Taking into account the particular and homogeneous components of $\boldsymbol{f}_c$, the constraints imposed by the friction cones can be expressed in a matricial form as:

$$\mathcal{R}\boldsymbol{h} + \boldsymbol{b} \leq \boldsymbol{0} \tag{14}$$

where $\mathcal{R} \in \mathbb{R}^{nm\times n}$ and $\boldsymbol{b} \in \mathbb{R}^{nm}$. As a result, the problem of minimizing the grasping forces can be expressed as the following minimization problem:

$$\text{Min} \quad \mathcal{F}(\boldsymbol{h}) = \|\boldsymbol{h}\|^2 \tag{15}$$

$$\text{subject to} \quad \mathcal{R}\boldsymbol{h} + \boldsymbol{b} \leq \boldsymbol{0} \tag{16}$$

Note that Eq. (13) and (15) are equivalent for the optimization and they give the same result. Besides, these equations are quadratic implying a non-linear minimization problem. From Eq. (10), the Hessian matrix associated to this problem is $\mathcal{H}(\boldsymbol{h})=2\mathcal{I}_n$, $\mathcal{I}_n$ being the n-identity matrix. Based on $\mathcal{H}(\boldsymbol{h})$ the following properties regarding the primal and dual forms are stated:

- $\mathcal{H}(\boldsymbol{h})$ is constant and positive definite in all the subspace defined by the constraints. Therefore, the Kuhn-Tucker conditions determine the strictly global minimum of the problem.
- The convexity property necessary to apply the dual theorem is satisfied. Then, the solution of the dual form gives also the strictly global minimum of the primal form.

The following subsections present the primal and dual forms of the minimization problem described by Eq. (15) and (16).

3.2 Primal Form

Since the convexity of the minimization problem is assured, the global minimal grasping forces satisfy the following system of equations obtained from the Kuhn-Tucker conditions:

$$2\mathcal{I}_n\boldsymbol{h} + \mathcal{R}^T\boldsymbol{\lambda} = \boldsymbol{0} \tag{17}$$

$$\boldsymbol{\lambda}(\mathcal{R}\boldsymbol{h} + \boldsymbol{b}) = \boldsymbol{0} \tag{18}$$

$$\boldsymbol{\lambda} \geq \boldsymbol{0} \tag{19}$$

In this problem, the number of constraints (nm) is larger than the number of variables (n) implying that the maximum number of active constraints has to be n so that the solution is regular. Therefore, at least $nm - n$ Lagrange multipliers are equal to zero. The determination of the minimal grasping forces using the Kuhn-Tucker conditions represents a combinatorial problem with the maximum number of combinations bounded by C^n_{nm}. Each combination implies to solve a n-linear system of equations. Note that the number of combinations increases exponentially with respect to the number of faces used to linearize the friction cones. Although the Kuhn-Tucker conditions in their pure form can be used when the friction cones are linearized with a low number of faces, it should be used jointly with another methods to improve the convergence.

3.3 Dual Form

Since the convexity of the minimization problem is assured, the dual theorem can be applied obtaining that the solution of the optimization problem described by Eq. (15) and (16) is also the solution of the following maximization problem:

$$\text{Max} \quad \phi(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^T\mathcal{S}\boldsymbol{\lambda} + \boldsymbol{\lambda}^T\boldsymbol{b} \tag{20}$$

$$\text{subject to} \quad \boldsymbol{\lambda} \geq \boldsymbol{0} \tag{21}$$

where $\mathcal{S} \in \mathbb{R}^{n\times nm}$ is defined as:

$$\mathcal{S}(i,j) = \left\{ \begin{array}{ll} -\dfrac{1}{4}\displaystyle\sum_{k=1}^{n}[R(i,k)]^2 & \text{if } i=j \\ -\dfrac{1}{4}\displaystyle\sum_{k=1}^{n}[R(i,k)][R(j,k)] & \text{if } i \neq j \end{array} \right\} \tag{22}$$

The dual form has the following important advantages respect to the primal one:

1. The objective functions of the two forms are quadratic, but the constraints of the dual form are much simpler than the constraints of the primal form.
2. $\boldsymbol{\lambda} = \boldsymbol{0}$ satisfies these constraints, thus it is trivial to find an initial value $\boldsymbol{\lambda}$ inside the feasible regions. Besides, this is a good initial value since at least $nm - n$ Lagrange multipliers are equal to zero for the solution of the primal form to be regular.
3. The progress from the initial value to the optimal solution maintaining the partial results inside the feasible region is also a simple task.

Nevertheless, a drawback of this formulation is the lack of physical meaning. The accomplishment of the constraints does not have any physical meaning and it does not imply to satisfy the friction constraints. Only in the optimum case it is possible to assure that the finger forces lie inside the friction cones. Even so, it is considered that the mathematical advantages of the dual problem worth its lack of physical meaning. Fig. 2 schematizes the relation between the primal and the dual form.

Using the dual problem formulation, the following algorithm based on the gradient of the maximiza-

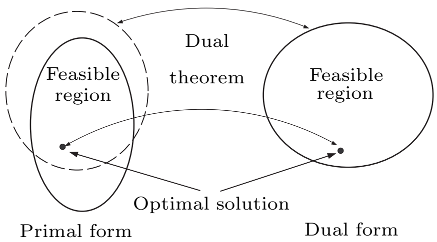

Fig. 2. Relations between the primal and the dual forms: the optimal solutions are equivalent while the feasible regions are not.

tion function described by Eq. (20) is applied to determine the optimal solution.

Step 1. *Initialization.*
Select as initial feasible solution $\boldsymbol{\lambda}^{(0)} = \mathbf{0}$, chose the step size α and the tolerance parameter $\varepsilon > 0$, compute $\nabla\phi$ and initialize $t = 1$.

Step 2. *Update values.*
Compute $\boldsymbol{\lambda}^{(t+1)} = \boldsymbol{\lambda}^{(t)} + \alpha\nabla\phi\boldsymbol{\lambda}^{(t+1)}$.
If $\lambda_i^{(t+1)} < 0$ then $\lambda_i^{(t+1)} = 0$, where $\lambda_i^{(t+1)}$, with $i = 1, ..., nm$, is a component of $\boldsymbol{\lambda}^{(t+1)}$.

Step 3. *Stop condition.*
Compute $\delta = \|\boldsymbol{\lambda}^{(t+1)} - \boldsymbol{\lambda}^{(t)}\|$.
If $\delta < \varepsilon$ then $\boldsymbol{\lambda}^{(t+1)}$ is the optimal solution of the dual problem. The optimal solution of the primal problem is determined using the Kuhn-Tucker conditions.
Else $t = t + 1$ and go to Step 2.

This algorithm uses a constant value for the step size α that has been chosen empirically (the smaller the magnitude of α the slower the convergence, but if α is too large the algorithm may never converge). In order to increase the convergence, more sophisticated methods that vary the step size as a function of the rate of convergence will be considered in future works. This methods could be easily introduced in the algorithm with minor modifications.

4. EXAMPLES

This section presents some results of applying the proposed methodology to compute the grasping forces given the positions of the contact points on the object boundary and the external wrench ($\boldsymbol{w}_{ext}$) exerted on it. The methodology has been implemented using Matlab 6.5 in a Pentium Centrino at 1.6 GHz. Therefore, the code is not optimal in terms of efficiency and the computational times included in the examples can only be considered as qualitative values. Fig. 3 shows the object and the three grasping points used in the examples. The position of the contact points with respect to the object reference frame of the object are given by the following grasp matrix:

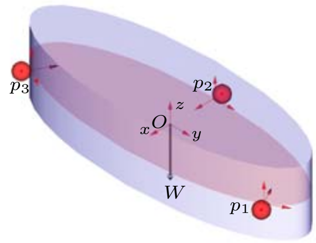

Fig. 3. Object and section defined by the three grasping points used in the examples.

$$\mathcal{G}=\begin{bmatrix} -0.86 & 0.50 & 0 & 1 & 0 & 0 & -0.86 & 0.50 & 0 \\ -0.50 & -0.86 & 0 & 0 & 1 & 0 & 0.50 & 0.86 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2.59 & 0 & 0 & 0 & 0 & 0 & -2.59 \\ 0 & 0 & -0.50 & 0 & 0 & 1 & 0 & 0 & -0.50 \\ 2 & -1.73 & 0 & 0 & -1 & 0 & -2 & 1.73 & 0 \end{bmatrix}$$

Three examples taking into account different values of $\boldsymbol{w}_{ext}$ will be presented. The following parameters are used in the three examples: $\mu = 0.3$ (friction coefficient), $m = 12$ (number of faces used to linearize the friction cones), $\alpha = 0.7$ (step size of the gradient algorithm) and $\varepsilon = 1 \cdot 10^{-4}$ (tolerance used in the stop condition).

Example 1. Only the object weight acts as external wrench and the object is positioned as in Fig. 3. Therefore, $\boldsymbol{w}_{ext} = [0\ \ 0\ -1\ \ 0\ \ 0\ \ 0]^T$.

The gradient method based on the dual problem formulation finds the optimal solution at step 730 ($\sim$ 20 ms). The final value of the objective function is $\|\boldsymbol{f}_c\| = 2.4867$ and the finger forces are: $\boldsymbol{f}_{c_1} = [1.11\ \ 0.09\ \ 0.33]^T$, $\boldsymbol{f}_{c_2} = [1.83\ \ 0\ \ 0.33]^T$ and $\boldsymbol{f}_{c_3} = [1.11\ \ 0.09\ \ 0.33]^T$.

Note that the grasp matrix $\mathcal{G}$ has been defined such that the third component of each finger force is orthogonal to the plane defined by the three contact points (shaded plane in Fig. 3). The external force is also orthogonal to this plane, then, it is balanced by the third components of the forces while the other components are due to satisfy the friction constraints.

Example 2. Now the external applied wrench has non-null components in each direction of the object reference frame: $\boldsymbol{w}_{ext} = [1\ -2\ \ 5\ -4\ \ 1\ \ 2]^T$. The gradient method based on the dual problem formulation finds the optimal solution at step 888 ($\sim$ 20 ms). The final value of the objective function is $\|\boldsymbol{f}_c\| = 12.5725$ and the finger forces are: $\boldsymbol{f}_{c_1}$=[4.12 –1.10 –0.56]^T, $\boldsymbol{f}_{c_2}$=[8.91 –0.89 –2.33]^T and $\boldsymbol{f}_{c_3} = [7.01\ \ 0.56\ -2.10]^T$.

Example 3. In this example the object is rotated clockwise 2π radians respect to the z-axis (Fig. 4 shows the initial position of the object). The forces are computed while the object is being rotated since the direction of the weight with respect to the contact points varies. The movement has been discretized with 100 sampling points, which implies that the forces are recomputed at each

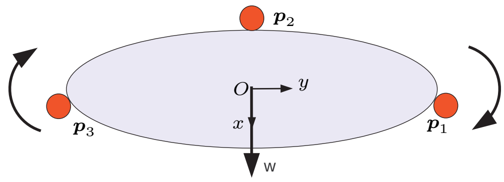

Fig. 4. Initial position of the object considered in example 3.

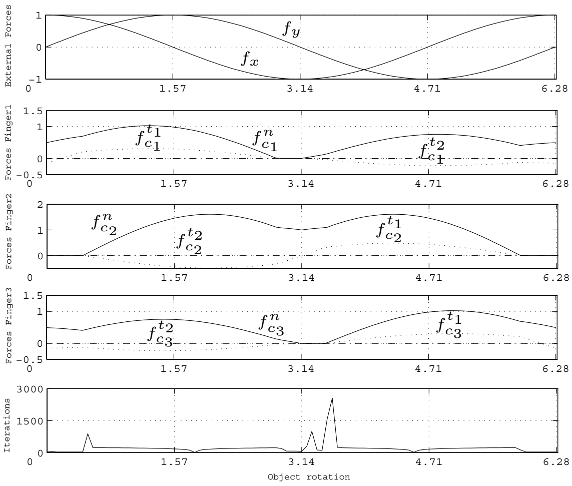

Fig. 5. Evolution of the finger forces during the object rotation in example 3.

0.0628 rad. In order to improve the efficiency of the gradient method, $\boldsymbol{\lambda}$ is initialized with the solution of the previous sampling point instead of the null vector. The average and maximum number of iterations to obtain the optimal solution at each step is 216 ($\sim 10\,ms$) and 2545 ($\sim 90\,ms$), respectively. Fig. 5 shows the evolution of the finger forces during all the movement.

5. CONCLUSIONS AND FUTURE WORKS

A new mathematical approach to solve the force distribution problem in a grasp has been presented. This approach is based on the dual theorem of non-linear programming, which can only be applied when the convexity of the problem is assured. By adequately modeling the problem and applying the dual theorem, the original problem is transformed to another one much easier to solve. The examples shows the efficiency of the proposed methodology. Even when the code can not be considered optimal in terms of efficiency, the provided computational times are of the order or even smaller than those of some of the most popular algorithms described by Liu and Li (2004).

The improving of the proposed gradient algorithm considering a variable step size as a function of the rate of convergence is considered as future work. Another interesting future work is the study of the problem convexity considering non-linear constraints in order to apply the dual theorem in this case.

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