Fast and Flexible Determination of Force-Closure Independent Regions to Grasp Polygonal Objects*

Jordi Cornellà Raúl Suárez
Institut d'Organització i Control de Sistemes Industrials (IOC-UPC)
Av. Diagonal 647 Planta 11, 08028 Barcelona, SPAIN
Emails: jordi.cornella@upc.es, raul.suarez@upc.es

***Abstract*— Force-closure independent regions are parts of the object edges such that a grasp with a finger in each region ensures a force-closure grasp. These regions are useful to provide some robustness to the grasp in the presence of uncertainty as well as in grasp planning. Most of the approaches to the computation of these regions for N fingers work on the contact space, implying a N-dimensional problem. This paper presents a new approach to determine independent regions on polygonal objects considering N friction or frictionless contacts. The approach works on the object space, implying that it is always a two-dimensional problem and, since it is not necessary to compute all the force-closure space, it becomes a very fast approach. Besides, the approach is also flexible since constraints on the fingers placement can be easily introduced. Some graphical examples are included in the paper showing the simplicity of the methodology.**

***Index Terms*— Grasp synthesis, force/form-closure, force-closure independent regions.**

I. Introduction

The obtention of grasps capable of ensuring the immobility of the object despite external disturbances has been a topic of great interest in grasping and manipulation of objects. These grasps are characterized by the properties of form-closure or force-closure [1]. In order to select a grasp among all the possible force-closure grasps (hereafter FC grasps), algorithms that optimize a quality criterion (for instance [2] [3] [4]) or algorithms based on heuristics criterions (for instance [5] [6]) were developed. These algorithms determine "precision" grasps, i.e. grasps formed by a set of contacts points on the object where the fingertips will be placed, and they require a good precision in the fingertip placements (in [2] and [3] the robustness in front of the finger positioning errors is partially treated). In a real execution, the final grasp and the theoretical grasp may differ due to fingers positioning errors. A metric for measuring the sensitivity of a grasp to positioning errors can be found in [7]. In order to provide robustness to the grasp in front of these errors, Nguyen [8] introduced the concept of independent regions, i.e. regions on the object boundary such that a finger in each region ensures a FC grasp independently of the exact contact point, and he developed a geometrical approach to determine the maximum independent regions on polygonal objects using four frictionless contacts and two friction contacts. The problem of determining independent regions using four frictionless contacts was also treated in [9]. Ponce and Faverjon [10] and Ponce et al. [11] extended Nguyen's approach to three finger grasps on polygonal objects and to four finger grasps on polyhedral objects, respectively. Liu [12] and Li, Yu and Tsujio [13] proposed algorithms to determine all the N-finger FC grasps on polygonal objects. These algorithms have not been used to compute independent regions, although in [13] the most stable grasp considering fingers positioning errors is determined. Recently, Pollard [14] presented an approach to determine independent regions on 3D objects based on initial examples, but the selection of a good initial example for a given object is a critical step.

This paper deals with the problem of determining independent regions on polygonal objects considering N friction or frictionless contacts. Since the space defined by all the FC grasps may be concave (this result can be obtained from [12] and [13] and it will be also shown here) it is decomposed into a set of convex subspaces, establishing each one a necessary and sufficient condition for the existence of a FC grasp and, in order to obtain it, at least one of these conditions must be satisfied. The computational cost of the decomposing algorithm for N fingers is $O(N^3)$. A condition to determine independent regions on each subspace is also presented and, using its geometrical interpretation, the problem of determining independent regions is reduced to find two particular points on the object space. The approach is very fast since it is applied on the object space and it is not necessary to compute the N-dimensional space of all the FC grasps, whose computational cost is at least $O(N^3 \log N)$. Besides, other constraints on the fingers placement can be easily introduced, providing flexibility to the algorithm. The approach developed here follows a previous work of the authors [15] where linear programming was used to obtain maximum independent region on the contact space, while in this paper a faster approach is presented. In this work it is assumed that: the edges of the polygon where the fingers will contact are given; the forces applied by the fingers act only against the object boundary; and the fingertip is a point. Note that in this approach there is no constraint regarding the number of fingers neither the number of fingers per edge.

II. Wrench Space

A. Representation of forces and torques

Let $\boldsymbol{f}_i = \alpha_i \widehat{\boldsymbol{f}}_i$ be the force exerted by each finger i on the object boundary at each contact point with $\alpha_i \geq 0$ and $\|\widehat{\boldsymbol{f}}_i\| = 1$. In the absence of friction, $\widehat{\boldsymbol{f}}_i$ is normal to the object boundary, i.e. $\widehat{\boldsymbol{f}}_i = [\cos\theta_i \ \sin\theta_i]$ where θ_i indicates

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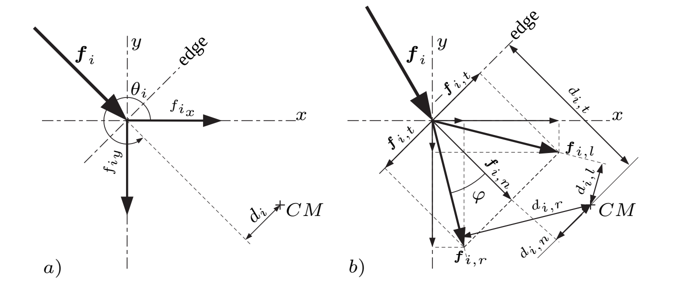

Fig. 1. *a) Frictionless contact; b) Friction contact, where $\boldsymbol{f}_{i,l}$ and $\boldsymbol{f}_{i,r}$ are the primitive forces, $\boldsymbol{f}_{i,n}$ is the normal force and $\boldsymbol{f}_{i,t}$ is the tangential force.*

the inward direction normal to the contact edge. The force exerted by each finger produces a torque τ_i with respect to the object's center of mass (CM), and the components of $\boldsymbol{f}_i$ and τ_i form the wrench vector $\boldsymbol{\omega}_i = [\boldsymbol{f}_i \ \tau_i]^T$ (see Fig. 1.a).

When friction is taken into account, $\boldsymbol{f}_i$ can be decomposed in two components $\boldsymbol{f}_{i,n}$ and $\boldsymbol{f}_{i,t}$ which are respectively normal and tangent to the contact edge (see Fig. 1.b). In order to avoid that the finger slips on the edge, the Coulomb's law must be accomplished: $\|\boldsymbol{f}_{i,n}\| \geq \mu \|\boldsymbol{f}_{i,t}\|$, where μ is the friction coefficient. This implies that $\boldsymbol{f}_i$ can be applied in a range of directions around the normal of the contact edge, determining the friction cone. Then, $\boldsymbol{f}_i$ can be expressed as the summation of two forces, usually called primitive forces, as $\boldsymbol{f}_i = \boldsymbol{f}_{i,l} + \boldsymbol{f}_{i,r}$. The primitive forces produce the primitive torques $\tau_{i,l}$ and $\tau_{i,r}$, forming the primitive wrenches $\boldsymbol{\omega}_{i,l} = [\boldsymbol{f}_{i,l} \ \tau_{i,l}]^T$ and $\boldsymbol{\omega}_{i,r} = [\boldsymbol{f}_{i,r} \ \tau_{i,r}]^T$, and the wrench produced by $\boldsymbol{f}_i$ is the summation of them, i.e. $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{i,l} + \boldsymbol{\omega}_{i,r}$.

Besides, the primitive torques can be expressed as the summation of the torques produced by the components of the primitive forces normal and tangential to the edge as $\tau_{i,r} = \tau_{i,n} + \tau_{i,t}$ and $\tau_{i,l} = \tau_{i,n} - \tau_{i,t}$, where $\tau_{i,n}$ and $\tau_{i,t}$ are the torques produced by $\boldsymbol{f}_{i,n}$ and $\boldsymbol{f}_{i,t}$, respectively. Thus, the relation between two primitive torques is

$$\tau_{i,r} = \tau_{i,l} + 2\tau_{i,t} \tag{1}$$

B. Constraint on the finger forces

The forces applied by the fingers can be subject to different constraints [16]. The constraint considered in this work is that the total force exerted by the N fingers is limited, i.e. $\sum_{i=1}^{N} \alpha_i \leq \alpha_{max}$, for instance, due to a maximum available power for all the finger actuators. Then, the applied forces can generate a resultant force

$$\boldsymbol{f} = \sum_{i=1}^{N} \boldsymbol{f}_i = \alpha \widehat{\boldsymbol{f}} \tag{2}$$

Geometrically, this constraint implies that the fingers can apply forces on the object that produce a resultant inside the polygon:

$$\mathcal{P}_f = \textit{ConvexHull}\Big(\bigcup_{i=1}^{N} \{\boldsymbol{f}_i\}\Big) \quad \text{with } \boldsymbol{f}_i = \alpha_{max} \widehat{\boldsymbol{f}}_i \tag{3}$$

and the resultant possible applied wrenches are those inside the polyhedron $\mathcal{P}_1$ defined in the wrench space as:

$$\mathcal{P}_1 = \textit{ConvexHull}\Big(\bigcup_{i=1}^{N} \{\boldsymbol{\omega}_i\}\Big) \quad \text{for } \boldsymbol{f}_i = \alpha_{max} \widehat{\boldsymbol{f}}_i \tag{4}$$

In the rest of the paper, for simplicity and without loss of generality, we consider $\alpha_{max} = 1$ and $\boldsymbol{f}_i$ will refer always to the maximum (unitary) possible applied force.

A FC grasp must satisfy $\mathbf{0} \in \mathcal{P}_1$ [17]. Note $\mathcal{P}_f$ is the projection of $\mathcal{P}_1$ on the force space (i.e. the 2D subspace of the wrench space defined by pure forces) and it can be determined knowing the contact edges. Then, in the rest of the paper it is considered that $\mathbf{0} \in \mathcal{P}_f$ is satisfied.

III. Force-closure space

Definition 1: The *contact space* is the space defined by N paramaters that represent the grasping contact points on some given edges of an object. ◇

Given the contact edges there is a univocal relation between the torques produced by the unitary normal forces and primitive forces and the exact contact point. Thus, the parameters used in this paper to define the contact space are the torques produced by unitary normal forces when frictionless contacts are considered and the torques produced by the unitary primitive forces (related to each other by eq. (1)) when friction contacts are considered.

Definition 2: The *force-closure space, FC-space,* is the subset of the contact space where FC grasps can be obtained. ◇

A methodology to obtain the FC-space as the union of a set of convex subspaces is presented in this section. The obtained result is similar to the result in [13], although the initial considerations are different (in [13] there is not any constraint on the finger forces). Besides, the approach developed here determine additional information on the finger forces that is quite useful in the determination of the independent regions.

A. Convex FC-subspace using four frictionless contacts

Definition 3: The *Real Range* of τ_i, R_i, is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ that are physically possible due to the length of the contact edge. ◇

Definition 4: The *Directional Range* of τ_i, R_{fc_i}, is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ that allow a FC grasp considering that the contact edge has infinite length (i.e. only the "direction" of the edge is considered) for any other given three τ_j. ◇

From these two definitions, the existence of a FC grasp implies that $R_i \cap R_{fc_i} \neq \emptyset$. Since R_i is known, the valid torques that produce a FC grasp can be determined by finding R_{fc_i}.

Four frictionless contacts generate the minimum number of wrenches necessary to obtain a FC grasp [18], generating a convex hull $\mathcal{P}_1$ with minimum number of faces. Since a FC grasp must satisfy $\mathbf{0} \in \mathcal{P}_1$ [17] and $\mathcal{P}_1$ is convex, the Directional Range R_{fc_i}, $i = 1, ..., 4$, is a continuous set that has one or two finite extremes. Then, the type of Directional Range can be:

Infinite: R_{fc_i} has only one finite extreme τ_{i_m}. Then, $R_{fc_i} = [\tau_{i_m}, \infty)$ or $R_{fc_i} = (-\infty, \tau_{i_m}]$.

Limited: R_{fc_i} have two finite extremes τ_{i_m} and $\tau_{i_{m'}}$. Then, $R_{fc_i} = [\tau_{i_m}, \tau_{i_{m'}}]$ or $R_{fc_i} = [\tau_{i_{m'}}, \tau_{i_m}]$.

Proposition 1: Consider four applied contact forces $\boldsymbol{f}_i$, $i = 1, ..., 4$. The number of finite extremes and, therefore, the type of the Directional Range R_{fc_i} can be determined knowing how many pairs $\beta_{i,jk}$ and $\beta_{i,kj}$ are non-positive, being:

$$\beta_{i,jk} = \frac{\sin(\theta_i - \theta_k)}{\sin(\theta_j - \theta_k)} \quad (5)$$

$$\beta_{i,kj} = \frac{\sin(\theta_j - \theta_i)}{\sin(\theta_j - \theta_k)} \quad (6)$$

where θ_i, θ_j and θ_k are the directions of $\boldsymbol{f}_i$, $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$. ◇

Proof: Let τ_{i_m} be an extreme of R_{fc_i}. Since $\mathcal{P}_1$ is convex, if τ_{i_m} defines a vertex of $\mathcal{P}_1$, then $\mathbf{0} \in \partial\mathcal{P}_1$, $\partial\mathcal{P}_1$ being the boundary of $\mathcal{P}_1$. Thus, $\mathbf{0}$ can be expressed as a positive linear combination of the three vertices that define the face of $\partial\mathcal{P}_1$ containing $\mathbf{0}$, i.e. $\mathbf{0} = \gamma_i \boldsymbol{\omega}_{i_m} + \gamma_j \boldsymbol{\omega}_j + \gamma_k \boldsymbol{\omega}_k$ with $\gamma_i, \gamma_j, \gamma_k \geq 0$ and $\gamma_i + \gamma_j + \gamma_k = 1$. Solving this expression for $\boldsymbol{\omega}_{i_m}$ results

$$\cos\theta_i = \beta_{i,jk}\cos\theta_j + \beta_{i,kj}\cos\theta_k \quad (7)$$

$$\sin\theta_i = \beta_{i,jk}\sin\theta_j + \beta_{i,kj}\sin\theta_k \quad (8)$$

$$\tau_{i_m} = \beta_{i,jk}\tau_j + \beta_{i,kj}\tau_k \quad (9)$$

with $\beta_{i,jk} = -\frac{\gamma_j}{\gamma_i} \leq 0$, $\beta_{i,kj} = -\frac{\gamma_k}{\gamma_i} \leq 0$ and non simultaneously null. Then, there are three equalities with only two unknowns, $\beta_{i,jk}$ and $\beta_{i,kj}$.

If the forces $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ are not parallel, equalities (7) and (8) are independent, and the two unknowns $\beta_{i,jk}$ and $\beta_{i,kj}$ can be obtained from them as eq. (5) and (6). If $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$ are parallel (i.e. $\theta_j = \theta_k + t\pi$, $t \in \{0, 1\}$, meaning that there are two fingers on the same edge or on parallel edges) equalities (7) and (8) have no solution for $\beta_{i,jk}$ and $\beta_{i,kj}$, and therefore, from eq. (9), τ_{i_m} does not exist. Since no more than two forces can have the same direction in a four frictionless FC grasp, a minimum of one extreme always exist.

As a result, the number of extremes of R_{fc_i} are determined from the solutions of eq. (5) and (6) and therefore using information related only with the applied forces without taking into account the values of the torques. ◇

Proposition 2: Given four applied forces, the number of Infinite and Limited Directional Ranges are:

General case: If all the angles between the applied forces are different from π, there are two Infinite and two Limited Directional Ranges (Fig. 2a).

Particular cases: If the angle between two forces is π, there are three Infinite and one Limited Directional Ranges (Fig. 2b), and if the angles between two pairs of forces are π, the four Directional Ranges are Infinite (Fig. 2c). ◇

Proof: From Proposition 1 the type of Directional Range of τ_i is determined by the number of pairs of coefficients $\beta_{i,jk}$ and $\beta_{i,kj}$ that are non-positive, for $\{i, j, k\} \in \{1, 2, 3, 4\}$ and $i \neq j \neq k$. From eq. (5) and (6), the coefficients $\beta_{i,jk}$ and $\beta_{i,kj}$ depend on the directions of three applied forces, which also define the coefficients $\beta_{j,ik}$ and $\beta_{j,ki}$, and $\beta_{k,ji}$ and $\beta_{k,ij}$, with

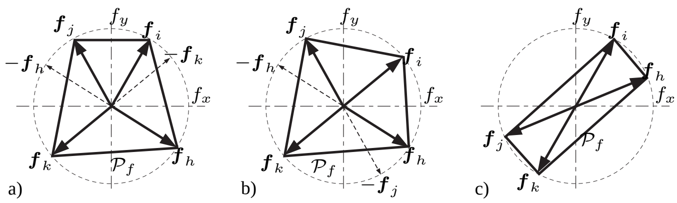

Fig. 2. *Examples of the determination of the types of Directional Ranges from the applied forces: a) General case: R_{fc_i} and R_{fc_j} are Infinite and R_{fc_h} and R_{fc_k} are Limited; b) Particular case: R_{fc_k} is Limited and R_{fc_h}, R_{fc_i} and R_{fc_j} are Infinite; c) Particular case: the four Directional Ranges are Infinite.*

the following relations between them

$$\beta_{i,jk} = \frac{1}{\beta_{j,ik}} = -\frac{\beta_{k,ji}}{\beta_{k,ij}} \quad (10)$$

$$\beta_{i,kj} = -\frac{\beta_{j,ki}}{\beta_{j,ik}} = \frac{1}{\beta_{k,ij}} \quad (11)$$

These relations imply that if one pair of coefficients is non-positive so are the other two pairs, determining one extreme for R_{fc_i}, R_{fc_j} and R_{fc_k}. Then, a valid subset of three forces generates three extremes. Since each force appears in three from the four possible subsets of forces, two subsets of forces are enough to determine all the extremes, otherwise there would be a Directional Range with more than two extremes or without any extreme, which is not possible.

In the general case, the two subsets of forces generate six different finite extremes for the four Directional Ranges. Therefore, there are two Directional Ranges with two extremes (so they are Limited), and two Directional Ranges with one extreme (so they are Infinite). In the particular case that the angle between two forces is π, the two subsets of forces generate five different finite extremes. Therefore, there are one Directional Range with two finite extremes (so it is Limited), and three Directional Ranges with one extreme (so they are Infinite). In the particular case that the angle between two pairs of forces is π, the two subsets of forces generate four different finite extremes. Therefore, the four Directional Ranges have one finite extreme (so they are Infinite). ◇

Knowing the directions of four applied forces, it can be easily identified which torques have Limited and which ones have Infinite Directional Ranges (it can be checked from eq. (5) and (6)): in the general case, the two Infinite Directional Ranges correspond to the torques generated by the two forces that lie between the negated of the other two (as in Fig. 2a), and in the particular case that the angle between two forces is π, the three Infinite Directional Ranges correspond to the torques generated by the other two forces and the force that lies between them (as in Fig. 2b).

Lemma 1: Let R_{fc_i} and R_{fc_j} be two Infinite Directional Ranges with $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ defining two consecutive vertices of $\mathcal{P}_f$. If R_{fc_i} tends to $\pm\infty$ then R_{fc_j} tends to $\mp\infty$. ◇

Proof: Consider first the general case with two Infinite and two Limited Directional Ranges. Let R_{fc_k} be one of the two Limited Directional Ranges. It is not known a priori if $R_{fc_k} = [\tau_{k_1}, \tau_{k_2}]$ or $R_{fc_k} = [\tau_{k_2}, \tau_{k_1}]$, then the two cases

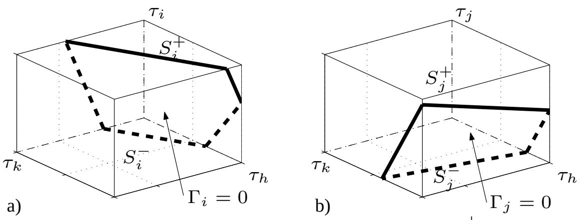

Fig. 3. *Polyhedrons resulting from the projection of: a) S_i^+ and S_i^- on the subspace $\{\tau_h,\tau_k,\tau_i\}$; b) S_j^+ and S_j^- on the subspace $\{\tau_h, \tau_k, \tau_j\}$.*

must be considered. If $R_{fc_k}=[\tau_{k_1},\tau_{k_2}]$ then $\tau_{k_1}\leq\tau_k\leq\tau_{k_2}$, and substituting τ_{k_1} and τ_{k_2} by their expressions derived from eq. (9), we obtain

$$\beta_{k,hj}\tau_h + \beta_{k,jh}\tau_j \leq \tau_k \leq \beta_{k,ih}\tau_i + \beta_{k,hi}\tau_h \qquad (12)$$

If τ_i and τ_j are solved from eq. (12), then

$$\tau_i \leq \frac{1}{\beta_{k,ih}}(\tau_k - \beta_{k,hi}\tau_h) \qquad (13)$$

$$\tau_j \geq \frac{1}{\beta_{k,jh}}(\tau_k - \beta_{k,hj}\tau_h) \qquad (14)$$

Therefore, τ_i has an upper bound while τ_j has a bottom bound implying that R_{fc_i} tends to $-\infty$ and R_{fc_j} tends to $+\infty$. If $R_{fc_k}=[\tau_{k_2},\tau_{k_1}]$ then, with the same reasoning, equations equivalent to (13) and (14) are obtained with swapped inequalities. Then, τ_i has a bottom bound while τ_j has an upper bound implying that R_{fc_i} tends to $+\infty$ and R_{fc_j} tends to $-\infty$.

The two particular cases can be tackle as limits of the general case. Adding $\delta\theta$ arbitrarily small to one of the aligned forces, the particular cases are transformed into the general case. Then, the above procedure can be applied obtaining the same results when $\delta\theta \rightarrow 0$. ⋄

From Lemma 1, the following necessary and sufficient condition for the existence of a FC grasp can be enunciated.

Necessary and sufficient condition (frictionless contacts). Four frictionless contacts allow a FC grasp if and only if

$$sign(\Gamma_i) \neq sign(\Gamma_j) \qquad (15)$$

with

$$\Gamma_\rho = \beta_{\rho,hk}\tau_h + \beta_{\rho,kh}\tau_k - \tau_\rho \qquad (16)$$

where $\rho\in\{i,j\}$, τ_i and τ_j have Infinite Directional Ranges and $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ define two consecutive vertices of $\mathcal{P}_f$ ⋄

Considering the Real Range of each torque, the geometrical interpretation of this necessary and sufficient condition is:

$$(S_i^+ \cap S_j^-) \cup (S_i^- \cap S_j^+) \neq \emptyset \qquad (17)$$

where S_ρ^+ and S_ρ^-, with $\rho\in\{i,j\}$ are the following polytopes:

$$S_\rho^+ = \{\{\tau_\rho,\tau_h,\tau_k\}|\tau_\rho\in R_\rho,\tau_h\in R_h,\tau_k\in R_k,\Gamma_\rho\geq 0\}$$
$$S_\rho^- = \{\{\tau_\rho,\tau_h,\tau_k\}|\tau_\rho\in R_\rho,\tau_h\in R_h,\tau_k\in R_k,\Gamma_\rho\leq 0\}$$

These polytopes can be represented as polyhedrons in two different 3-dimensional subspaces defined by $\{\tau_h,\ \tau_i,\ \tau_k\}$ and $\{\tau_h,\ \tau_j,\ \tau_k\}$, as in Fig. 3. By construction $S_i^+\cap S_j^-$ and $S_i^-\cap S_j^+$ are convex sets. Therefore, four applied forces determine two convex FC-subspaces.

Equation (16) has another useful geometrical property on the object space: the lines of action of $\boldsymbol{f}_\rho$, $\boldsymbol{f}_h$ and $\boldsymbol{f}_k$ intersect at the same point when $\Gamma_\rho = 0$. Grasps with this property was called critical grasps in [9] since it separates the FC grasps from the non-FC grasps. As a difference from [9], where it was considered that in the general case a critical grasp may be determined by any intersection of three lines of action of forces, the result obtained here determines exactly which are the forces whose lines of action intersect in a critical grasp.

B. Convex FC-subspaces using friction contacts.

Let $\boldsymbol{f}_{i,p}\in\{\boldsymbol{f}_{i,l},\boldsymbol{f}_{i,r}\}$ and $\tau_{i,p}\in\{\tau_{i,p},\tau_{i,l}\}$ be a primitive force and a primitive torque, respectively. Definitions 3 and 4 can be applied to friction contacts defining respectively $R_{i,p}$ and $R_{fc_{i,p}}$ for τ_{i_p}. The procedure developed to obtain the necessary and sufficient condition for frictionless contacts is based on the knowledge of four applied forces. Since the directions of the primitive forces are also known given the contact edges, the procedure developed in the previous subsection can also be applied with four primitive forces obtaining the following necessary and sufficient condition for friction contacts. Note that the two primitive torques of the same contact are not independent of each other and they must satisfy eq. (1) to obtain a real FC grasp.

Necessary and sufficient condition (friction contacts). Considering friction contacts, a FC grasp exists if and only if there are four primitive torques that satisfy the constraint of eq. (1) when two of them are from the same contact and

$$sign(\Gamma_i) \neq sign(\Gamma_j) \qquad (18)$$

with

$$\Gamma_\rho = \beta_{\rho,hk}\tau_{h,p} + \beta_{\rho,kh}\tau_{k,p} - \tau_{\rho,p} \qquad (19)$$

where $\rho\in\{i,j\}$, $\tau_{i,p}$ and $\tau_{j,p}$ have Infinite Directional Ranges and $\boldsymbol{f}_{i,p}$ and $\boldsymbol{f}_{j,p}$ define two consecutive vertices of $\mathcal{P}_f$ ⋄

The geometrical interpretation of this necessary and sufficient condition without considering eq. (1) is identical to the geometrical interpretation of the necessary and sufficient condition for frictionless contacts but, in this case, considering the primitive forces. Then, $(S_{i,p}^+\cap S_{j,p}^-)\cup(S_{i,p}^-\cap S_{j,p}^+)\neq\emptyset$. Equation (1) is a 2-dimensional subspace of the contact space, and its intersection with $(S_{i,p}^+\cap S_{j,p}^-)$ and $(S_{i,p}^-\cap S_{j,p}^+)$ determines the convex FC-subspaces.

Considering friction contacts, a critical grasp is obtained when three primitive forces intersect at the same point [19], and it happens when eq. (19) gives $\Gamma_\rho=0$.

C. Decomposition of the FC-space

When there are more than four normal or primitive forces (i.e. more than four frictionless contacts or more than two friction contacts), there may be several convex FC-subspaces limited by critical grasps. In order to obtain the combinations of four normal or primitive forces that determine convex FC-subspaces the following algorithm is used (the algorithm is described using the nomenclature of friction contacts but it can also be applied considering frictionless contacts changing the primitive forces by normal forces).

Algorithm 1: Let n_f be the number of primitive forces. The following steps are applied for each combination of any two primitive forces $\boldsymbol{f}_{h,p}$ and $\boldsymbol{f}_{k,p}$:

1. Search $\boldsymbol{f}_{i,p}$ with directions $\theta_{i,p}$ making $\boldsymbol{f}_{i,p} \in [-\boldsymbol{f}_{h,p}, -\boldsymbol{f}_{k,p}]$ for $i=1,...,n_f$ and $i \neq h \neq k$.
3. Let n_i be the number of $\boldsymbol{f}_{i,p}$ that satisfy step 2:
   - 3.1. If $n_i \leq 1$ then discard this combination of forces.
   - 3.2. If $n_i > 1$ then two convex FC-subspaces are obtained with each combination of $\boldsymbol{f}_{h,p}$, $\boldsymbol{f}_{k,p}$ and any other two forces from the n_i obtained in step 2. ◇

As a result, all the convex FC-subspaces and the type of Directional Range of each torque are determined with a computational cost $O(N^3)$. The FC-space is the union of these subspaces and it may be concave.

IV. Independent regions

The independent regions on the object boundary define a N-parallelepiped fully contained in the FC-space. Then, the problem of determining the independent regions is equivalent to the problem of finding a N-parallelepiped fully contained in the FC-space. Since the FC-space may be concave, it is not possible to assure that a N-parallelepiped is fully contained in it just by testing if its vertices belong to the FC-space, as it is done in [10] (remember that in [10] a sufficient condition is used, therefore the whole FC-space is not considered). Using the convex FC-subspaces determined in the previous section and the following proposition (it is enunciated considering friction contacts) the problem can be solved just checking only two vertices of the parallelepiped.

Proposition 3: Consider a convex FC-subspace limited by the planes represented in eq. (19) when $\Gamma_\rho = 0$ for $\rho \in \{i, j\}$. The set $(\tau^-_{\nu,p}, \tau^+_{\nu,p})$, with $\nu \in \{h, i, j, k\}$, is an independent region in the Directional Range R_{fc_ν} if any two primitive torques of the same contact satisfy eq. (1), $\tau^-_{h,p}$, $\tau^-_{k,p}$ and $\tau^-_{\rho,p}$ make $\Gamma_\rho = 0$, and $\tau^+_{h,p}$, $\tau^+_{k,p}$ and $\tau^+_{\rho,p}$ make $\Gamma_\rho = 0$. ◇

Proof: The meaning of eq.(1) has been discussed in Section II, therefore only the other two conditions need to be proved here. Proposition 1 determines that the coefficients of eq. (19) are non-positive, implying that this equation represents two planes with negative slope when $\Gamma_\rho = 0$ for $\rho \in \{i, j\}$ (see Fig. 4).

If $\tau^+_{h,p}$, $\tau^+_{k,p}$ and $\tau^+_{\rho,p}$ are the maximum torques generated on their respective independent regions and they make $\Gamma_\rho = 0$ in eq. (19), then it is not possible to obtain other values of $\tau_{h,p}$, $\tau_{k,p}$ and $\tau_{\rho,p}$ that belong to the independent region (i.e., values smaller than $\tau^+_{h,p}$, $\tau^+_{k,p}$ that $\tau^+_{\rho,p}$) that make $\Gamma_\rho = 0$, because the slope of the plane represented in eq. (19) is negative. In the same way, if $\tau^-_{h,p}$, $\tau^-_{k,p}$ and $\tau^-_{\rho,p}$ are the minimum torques generate on their respective independent regions and they make $\Gamma_\rho = 0$ then it is not possible to obtain other values of $\tau_{h,p}$, $\tau_{k,p}$ and $\tau_{\rho,p}$ that belong to the independent region (i.e., values bigger than $\tau^-_{h,p}$, $\tau^-_{k,p}$ and $\tau^-_{\rho,p}$) that make $\Gamma_\rho = 0$, again because the slope of the plane represented in eq. (19) is negative. As a result, these two conditions assure that $\tau_{\nu,p} \in (\tau^-_{\nu,p}, \tau^+_{\nu,p})$, with $\nu \in \{h, i, j, k\}$, cannot make $\Gamma_\rho = 0$ implying that critical grasps can not belong to $(\tau^-_{\nu,p}, \tau^+_{\nu,p})$. Therefore, since the subspace is convex, these conditions determine the extremes of the independent regions. ◇

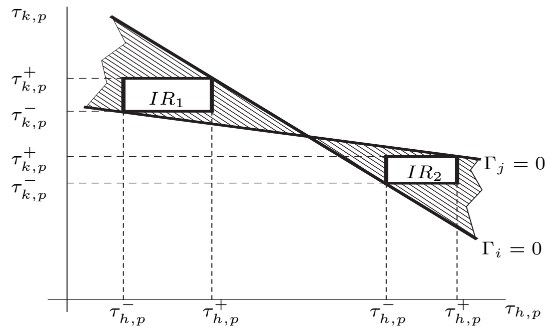

Fig. 4. *Two dimensional slice of a contact convex FC-subspace where $\tau_{h,p}$ and $\tau_{k,p}$ have Limited Directional Range and the other torques are constants. The parallelepipeds IR_1 and IR_2 satisfy Proposition 3 and they define two sets of independents regions.*

On the object space, $\Gamma_\rho = 0$ implies that the lines of action of $\boldsymbol{f}_{h,p}$, $\boldsymbol{f}_{k,p}$ and $\boldsymbol{f}_{\rho,p}$, intersect at the same point. As a consequence, Proposition 3 is equivalent to say that the lines of action of $\boldsymbol{f}_{h,p}$, $\boldsymbol{f}_{k,p}$ and $\boldsymbol{f}_{i,p}$ intersect at the same point when they are applied on the extremes of their respective independent regions where the maximum (or minimum) torque is produced, and the lines of action of $\boldsymbol{f}_{h,p}$, $\boldsymbol{f}_{k,p}$ and $\boldsymbol{f}_{j,p}$ intersect at the same point when they are applied on the extremes of their respective independent regions where the minimum (or maximum) torque is produced. Note that the independent regions determined according to Proposition 3 are defined on the Directional Ranges, so $(\tau^-_{\nu,p}, \tau^+_{\nu,p}) \cap R_{\nu,p} \neq \emptyset$, $\nu \in \{h, i, j, k\}$, must be satisfied to obtain the independent region on the real edge. The following algorithm is used to obtain the independent regions on the object edges:

Algorithm 2: Consider a convex FC-subspace limited by the planes represented in eq. (19) when $\Gamma_\rho = 0$ for $\rho \in \{i, j\}$.

1. Select two arbitrary points on the corresponding object edges where $\tau^+_{h,p}$ and $\tau^+_{k,p}$ will be produced.
2. Determine the point where $\tau^+_{i,p}$ is produced (the lines of action of the primitive forces applied on this point and on the two points selected in step 1 intersect at the same point). From eq. (19) $R_{fc_{i,p}}$ is Infinite, then it has only the finite extreme $\tau^+_{i,p}$.
3. Check if $R_{fc_{i,p}} \cap R_{i,p} \neq \emptyset$. If this condition is not satisfied a FC grasp is not possible. Then the independent regions do not exist for the initial selected points.
4. Determine the region where the intersection of the lines of action of $\boldsymbol{f}_{h,p}$, $\boldsymbol{f}_{k,p}$ and $\boldsymbol{f}_{j,p}$ must lie. This region must satisfy the following conditions: $\tau^+_{h,p} > \tau^-_{h,p}$, $\tau^+_{k,p} > \tau^-_{k,p}$, $R_{fc_{j,p}} \cap R_{j,p} \neq \emptyset$ and eq. (1). If these conditions are incompatible, then the independent regions do not exist for the initial selected points.
5. Select an arbitrary point in this region and project it on the edges, obtaining the extremes where $\tau^-_{h,p}$, $\tau^-_{k,p}$ and $\tau^-_{j,p}$ are produced.
6. Intersect the current independent regions with the real edges to obtain the actual independent regions (by construction these intersections are always not null). ◇

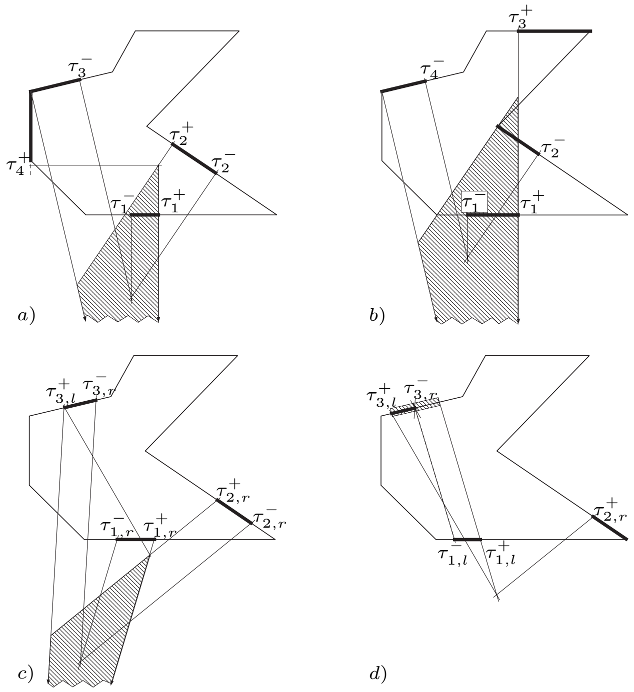

Fig. 5. *Examples of the determination of independent regions (black segments) on the edges of an object and the region determined in step 4 of algorithm 2 (shaded region): a) Four frictionless contacts where R_{fc_1} and R_{fc_2} are Limited and R_{fc_3} and R_{fc_4} are Infinite; b) Four frictionless contacts where R_{fc_1} is Limited and R_{fc_2}, R_{fc_3} and R_{fc_4} are Infinite; c) Three friction contacts where $R_{fc_{1,r}}$ and $R_{fc_{2,r}}$ are Limited and $R_{fc_{3,r}}$ and $R_{fc_{3,l}}$ are Infinite (the primitive forces considered in this case are $\boldsymbol{f}_{1,r}$, $\boldsymbol{f}_{2,r}$, $\boldsymbol{f}_{3,r}$ and $\boldsymbol{f}_{3,l}$); d) Three friction contacts where $R_{fc_{1,l}}$ and $R_{fc_{3,l}}$ are Limited and $R_{fc_{2,r}}$ and $R_{fc_{3,r}}$ are Infinite (the primitive forces considered in this case are $\boldsymbol{f}_{1,l}$, $\boldsymbol{f}_{3,l}$, $\boldsymbol{f}_{2,r}$ and $\boldsymbol{f}_{3,r}$ and the region is the shaded part of the contact edge).*

In analogous way, the algorithm can be applied selecting in the first step $\tau^-_{h,p}$ and $\tau^-_{k,p}$. Criteria to select the points in steps 1 and 5 are not discussed here, but it can be done considering other constraints on the finger positioning (for instance, kinematics constraints or task requirements), providing flexibility to the algorithm.

Proposition 3 and Algorithm 2 are also applicable considering frictionless contacts, exchanging the primitive forces for the normal forces and without considering eq. (1). Figure 5 shows examples considering friction and frictionless contacts, where it can be checked that it is not possible to intersect the lines of action of three forces when they are applied on independent regions, so critical grasps are not possible.

V. Conclusions and Future Works

In this paper a new approach to determine independent regions on 2D polygonal objects that allow a FC grasp considering any number of fingers has been presented. Since the FC-space may be concave, it is decomposed in a set of convex FC-subspaces establishing each one a necessary and sufficient condition for the existence of a FC grasp. A condition to obtain independent regions in each FC-subspace is also presented and using its geometrical interpretation the problem of determining independent region is reduced to find two particular points on the object space. The main advantages of the proposed algorithm are that it is applied on the object space, therefore the problem is always two-dimensional, and that it is not necessary to compute the N-dimensional FC-space. Besides, the algorithm is flexible and other constraints on the finger positioning can be introduced. The simplicity of the methodology and the fact that the algorithm is applied on the object space encourage to extend this work to non-polygonal objects and to 3D objects as future works. The methodology presented here is based on the relative positions of the applied forces, idea that can be furthered exploited in future works.

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