Computation of Force-Closure Independent Regions through Grasp Decomposition *

J. Cornellà, R. Suárez
Institut d'Organització i Control de Sistemes Industrials (IOC-UPC)
Av. Diagonal 647 Planta 11, 08028 Barcelona, SPAIN
Emails: jordi.cornella@upc.es, raul.suarez@upc.es

Abstract

The paper deals with the problem of determining independent regions on the edges of 2D polygonal objects that allow a force-closure grasp using N fingers. First, the grasp is decomposed into several non-redundant grasps, establishing each one a necessary and sufficient condition for force-closure. If at least one of them is satisfied, then the global grasp is also force-closure. The non-redundant grasps are used to determine the independent regions considering each condition on the given edges. The main advantage of the proposed approach is that it is not necessary to compute the entirely force-closure N-dimensional space for determining the independent regions. The algorithm has been implemented and a numerical example is included in the paper.

1 Introduction

Grasps capable of resisting external disturbances satisfy one of the following properties: form-closure (the position of the fingers ensures the object immobility) or force-closure (the forces applied by the fingers ensure the object immobility) [1]. The study of force-closure grasps, hereafter FC grasps, has been a topic of great interest in grasping and manipulation of objects. Mishra [7] enunciated a necessary and sufficient condition that a FC grasp has to satisfy. Based on this condition, some qualitative tests (determining if a set of contact points on the object allows a FC grasp) [2] and quantitative tests (measuring the goodness of the configuration) [4] have been proposed. The synthesis of optimal FC grasps based on the measure used in [4] was addressed in [3]. Another related problem is the determination of independent regions on the object boundary such that a finger in each region ensures a FC grasp. Nguyen [9] determine the maximum independent regions on 2D objects using a geometrical approach for four frictionless contacts and for two friction contacts. Ponce and Faverjon [10] determined these regions for three friction contacts using linear programming. Even when this approach has been extended to polyhedral objects [11], it is based on a sufficient condition and it only can evaluate a sub-set of all possible FC grasps. Moreover, the two works mentioned above ([9] and [10]) are specific for a given number of fingers. Liu [5] proposed an algorithm to determine the N-dimensional space that allow a FC grasp using N fingers on 2D objects. This is the same objective of [6], although in the latter without considering any constraint on the finger forces. Even when these two approaches are general and all the FC grasps are considered, they have not been used to compute the maximum independent regions.

This paper deals with the problem of determining the maximum independent regions on 2D objects, considering all the possible FC grasps. The approach developed here is based on a decomposition of any grasp (with any number of fingers) into several non-redundant grasps (with four frictionless contacts or two friction contacts). Each one of these non-redundant grasps has associated a necessary and sufficient condition for the existence of a FC grasp and, in order to produce a global FC grasp, at least one of them has to be satisfied. Then, the necessary and sufficient conditions are used independently in an algorithm based on linear programming to calculate the maximum independent regions on the object. The computational cost of the decomposing algorithm for N fingers is $O(N^3)$. The main advantage of this decomposition is that it is not necessary to compute the N-dimensional set of points that allow a FC grasp, whose computational cost is at least $O(N^3 \log N)$. Moreover, this set can be concave, making difficult the application of standard optimization algorithms.

The main assumptions considered in this work are: *1)* Grasped objects are planar and polygonal-shaped; *2)* The object edges where the fingers will contact are given; *3)* Forces applied by the fingers act only against the object boundary; *5)* The fingertip is a point. Note that in this approach there is no constraint regarding the number of fingers per edge, then, it is possible to consider two fingers on the same edge.

2 Wrench Space

2.1 Representation of forces and torques

Let $\boldsymbol{f}_i$ be the maximum force exerted by each finger on the object boundary at each contact point. In the absence of friction, $\boldsymbol{f}_i$ is the applied force normal to the object boundary and it produces a torque τ_i with respect to the object's center of mass. Considering

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that $\boldsymbol{f}_i$ is normalized to be $\|\boldsymbol{f}_i\| = 1$, the components of $\boldsymbol{f}_i$ (respect to the reference frame of the object) and τ_i form the wrench vector:

$$\boldsymbol{\omega}_i = [\cos\theta_i \quad \sin\theta_i \quad \tau_i]^T \tag{1}$$

where θ_i indicates the direction of $\boldsymbol{f}_i$. Since $\|\boldsymbol{f}_i\| = 1$, τ_i is equal to the distance d_i from the object's center of mass (CM) to the line of action of $\boldsymbol{f}_i$ (see Fig. 1.a).

When friction is taken into account, $\boldsymbol{f}_i$ can be decomposed in two components $\boldsymbol{f}_{i,n}$ and $\boldsymbol{f}_{i,t}$ which are respectively normal and tangent to the contact edge (see Fig. 1.b). In order to avoid that the finger slips on the edge, the Coulomb's law must be accomplished: $|\boldsymbol{f}_{i,n}| \geq \mu|\boldsymbol{f}_{i,t}|$, where μ is the friction coefficient. This implies that $\boldsymbol{f}_i$ can be applied in a range of directions around the normal of the contact edge, determining the friction cone. Then, $\boldsymbol{f}_i$ can be expressed as a positive linear combination of two forces:

$$\boldsymbol{f}_i = \alpha_{i,l}\boldsymbol{f}_{i,l} + \alpha_{i,r}\boldsymbol{f}_{i,r} \tag{2}$$

where $\boldsymbol{f}_{i,l}$ and $\boldsymbol{f}_{i,r}$ are the forces along the boundaries of the friction cone, usually called primitive forces.

Then, the wrench produced in a contact point can be expressed as a linear combination of two primitive wrenches:

$$\boldsymbol{\omega}_i = \alpha_{i,l}\boldsymbol{\omega}_{i,l} + \alpha_{i,r}\boldsymbol{\omega}_{i,r} \tag{3}$$

with

$$\boldsymbol{\omega}_{i,r} = [\cos(\theta_i - \varphi) \quad \sin(\theta_i - \varphi) \quad \tau_{i,r}]^T \tag{4}$$

$$\boldsymbol{\omega}_{i,l} = [\cos(\theta_i + \varphi) \quad \sin(\theta_i + \varphi) \quad \tau_{i,l}]^T \tag{5}$$

where θ_i indicates the direction normal to the edge i, $\varphi = \arctan\mu$ and $\tau_{i,r}$ and $\tau_{i,l}$ are the torques produced by $\boldsymbol{f}_{i,r}$ and $\boldsymbol{f}_{i,l}$, respectively.

Considering that the torque component of each primitive wrench is produced by the corresponding force components tangential and normal to the edge (see Fig. 1.b), we obtain:

$$\tau_{i,r} = \tau_{i,n} + \tau_{i,t} \tag{6}$$

$$\tau_{i,l} = \tau_{i,n} - \tau_{i,t} \tag{7}$$

where $\tau_{i,n}$ and $\tau_{i,t}$ are the torques produced by $\boldsymbol{f}_{i,n}$ and $\boldsymbol{f}_{i,t}$, respectively.

2.2 Constraint on the finger forces

The forces applied by the fingers can be subject to different constraints [8]. The constraint considered in this work is that the total force exerted by all the fingers is limited, for instance, due to a maximum available power for all the finger actuators. Then, the applied forces can generate a resultant wrench

$$\boldsymbol{\omega} = \sum_{i=1}^{N} \alpha_i \boldsymbol{\omega}_i \qquad \text{with} \sum_{i=1}^{N} \alpha_i \leq 1 \tag{8}$$

where N is the number of contacts. If friction contacts are considered, then $\alpha_i = \alpha_{i,l} + \alpha_{i,r}$ is a linear approximation.

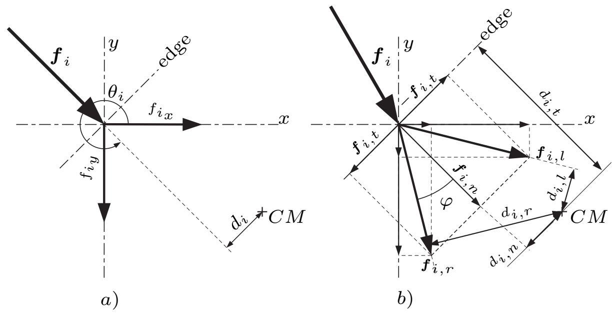

Figure 1: *a) Frictionless contact; b) Friction contact, where $\boldsymbol{f}_{i,l}$ and $\boldsymbol{f}_{i,r}$ are the primitive forces, $\boldsymbol{f}_{i,n}$ is the normal force and $\boldsymbol{f}_{i,t}$ is the tangential force.*

Geometrically, the resultant wrench can be any one inside the polyhedron $\mathcal{P}_1$ defined in the wrench space as:

$$\mathcal{P}_1 = \mathit{ConvexHull}(\bigcup_{i=1}^{N}\{\boldsymbol{\omega}_i\}) \tag{9}$$

3 Non-redundant grasps

Definition 1 A *non-redundant grasp*, is the grasp formed by the minimal number of contact points on the object that generate the necessary wrenches to obtain a FC grasp. ◇

FC grasps formed by four frictionless contacts or by two friction contacts are non-redundant grasps.

3.1 One unknown frictionless contact

This subsection summarizes part of the results obtained in our previous work [3] where, given three contact points on the object, the range of a given edge for the fourth finger that allow a FC grasp was determined using frictionless contacts.

Definition 2 The *nominal range* of τ_i, R_{fc_i}, is the range of values of τ_i that allow a FC grasp considering that edge i has infinite length (i.e. only the direction of the edge is taken account). ◇

In order to produce a FC grasp, $\mathcal{P}_1$ must contain the origin [7]. Then, in order for a value τ_i^* to be an extreme of R_{fc_i}, it is necessary and sufficient that it makes $0 \in \partial\mathcal{P}_1$, $\partial\mathcal{P}_1$ being the boundary of $\mathcal{P}_1$.

Let $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$ be three known wrenches (of three fingers already on the object) and $\boldsymbol{\omega}_4 = [f_{x_4} \ \ f_{y_4} \ \ \tau_4]^T$ the wrench whose component τ_4 is unknown, then the nominal range R_{fc_4} of τ_4 can be found with the following two steps:

1. Obtention of three candidates τ_{4_m} $(m = 1, 2, 3)$ to be possible extremes of R_{fc_4} from the intersection, in the wrench space, of the three planes defined by the sets of wrenches $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, 0\}$, $\{\boldsymbol{\omega}_1, \boldsymbol{\omega}_3, 0\}$ and $\{\boldsymbol{\omega}_2, \boldsymbol{\omega}_3, 0\}$, respectively, with the straight line determined by $x = f_{x_4}, y = f_{y_4}$.
2. Testing these candidates in the necessary and sufficient force-closure condition [3]:

$$\beta_{1,4_m}[f_{x_1} \ \ f_{y_1} \ \ \tau_1]^T + \beta_{2,4_m}[f_{x_1} \ \ f_{y_1} \ \ \tau_2]^T + \beta_{3,4_m}[f_{x_3} \ \ f_{y_3} \ \ \tau_3]^T = [f_{x_4} \ \ f_{y_4} \ \ \tau_{4_m}]^T \tag{10}$$

where $\beta_{1,4_m}$, $\beta_{2,4_m}$ and $\beta_{3,4_m}$ are the coefficients corresponding to the wrenches $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ and $\boldsymbol{\omega}_3$, respectively, used to obtain the candidate τ_{4_m}. These coefficients can be determined just from the knowledge of the applied forces and one of them is always null since the candidates to be extremes of R_{fc_i} are obtained as a function of only two other wrenches (step 1). If $\beta_{j,4_m} \leq 0$, $(j = 1, 2, 3)$ then the candidate τ_{4_m} is an extreme of R_{fc_4}.

Since $\mathcal{P}_1$ is convex, R_{fc_4} is a continuous set and only one or two of the three candidates can be valid extremes of R_{fc_4}. Depending on the number of valid extremes of R_{fc_4}, the type of nominal range is:

Infinite: if only one candidate τ_{4_m} satisfies the necessary and sufficient condition (here denoted as τ_{4_1}), then R_{fc_4} is the range determined by $R_{fc_4} = [\tau_{4_1}, \infty)$ or $R_{fc_4} = (-\infty, \tau_{4_1}]$ such that R_{fc_4} does not contain the other two τ_{4_m} with $m \neq 1$.

Limited: if two candidates τ_{4_m} satisfy the necessary and sufficient condition (here denoted as τ_{4_1} and τ_{4_2}) then $R_{fc_4} = [\tau_{4_1}, \tau_{4_2}]$ or $R_{fc_4} = [\tau_{4_2}, \tau_{4_1}]$.

3.2 Four unknown frictionless contacts

The approach used to determine the nominal range of one unknown contact point is extended here to the general case in which the four contact points are unknown (but the contact edges are known).

All the candidates to be extremes of R_{fc_i} with $i = 1, ..., 4$ are obtained from the torque component of equation (10) as:

$$\tau_{i_m} = \beta_{j,i_m} \tau_j + \beta_{k,i_m} \tau_k \tag{11}$$

where $\{i, j, k\} \in \{1, 2, 3, 4\}, i \neq j \neq k$, $m = \{1, 2, 3\}$, τ_{i_m} is the candidate m to be extreme of R_{fc_i} and β_{j,i_m} and β_{k,i_m} are the coefficients corresponding to the wrenches $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ used to obtain the candidate τ_{i_m}. β_{j,i_m} and β_{k,i_m} can be expressed in general form as:

$$\beta_{j,i_m} = \frac{\sin(\theta_i - \theta_k)}{\sin(\theta_j - \theta_k)} \tag{12}$$

$$\beta_{k,i_m} = \frac{\cos(\theta_i)\sin(\theta_j - \theta_k) - \cos(\theta_j)\sin(\theta_i - \theta_k)}{\cos(\theta_k)\sin(\theta_j - \theta_k)} \tag{13}$$

where θ_i, θ_j and θ_k are the angles that indicate the directions of $\boldsymbol{f}_i$, $\boldsymbol{f}_j$ and $\boldsymbol{f}_k$, respectively.

The signs of β_{j,i_m} and β_{k,i_m} depends only on the directions of the applied forces and they determine the number of valid extremes of R_{fc_i}. Then, the type of the nominal range R_{fc_i} can be automatically determined just from knowing the edge of the object to be contacted by each finger regardless of the actual position of the fingers on the edges.

Proposition 1 Consider the torques of the four wrenches that form a FC grasp. The nominal range of two of them are Limited and the nominal range of the other two are Infinite, except in the particular case in which the fingers are placed on parallel edges: if two fingers are placed in two parallel edges, then three nominal ranges are Infinite and the other one is Limited, and if the four fingers are placed in two pairs of parallel edges, then the four nominal ranges are Infinite. ◇

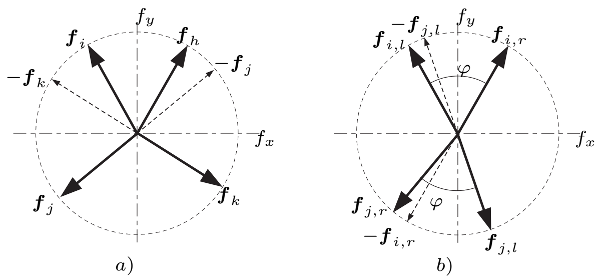

Figure 2: *Determination of the type of nominal range: a) Four frictionless contacts with R_{fc_h}, R_{fc_i} Infinite and R_{fc_j}, R_{fc_k} Limited; b)Two friction contacts with $R_{fc_{i,l}}$, $R_{fc_{j,r}}$ Infinite and $R_{fc_{j,l}}$, $R_{fc_{i,r}}$ Limited*

Proof: Considering all the combinations of subindexes in equation (11), four different equations are obtained. One of these equations with $\beta_{j,i_m}, \beta_{k,i_m} \leq 0$ defines an extreme of the three nominal ranges related with the forces that determine β_{j,i_m} and β_{k,i_m} (for instance, if $\beta_{2,1_1}, \beta_{3,1_1} \leq 0$ being determined by the forces $\boldsymbol{f}_1$, $\boldsymbol{f}_2$ and $\boldsymbol{f}_3$, then they define an extreme of R_{fc_1}, R_{fc_2} and R_{fc_3}). No more than two independent equations with $\beta_{j,i_m}, \beta_{k,i_m} \leq 0$ are possible, because otherwise one of the nominal ranges would have three extremes, and this is not possible since $\mathcal{P}_1$ is convex. Therefore, a total of six extremes are obtained for four nominal ranges, and then two of them are Limited (they have two extremes) and the other two are Infinite (they have only one extreme).

When two fingers are placed on two parallel edges, one of the coefficients of the equation that relates these two forces is always zero (i.e., the coefficient relating these two forces is always negative regardless the direction of the other forces). Then, this equation defines one extreme of only two nominal FC ranges, obtaining a total of five extremes. Then, three nominal ranges are Infinite (they have only one extreme) and the other one is Limited (it has two extremes). The same reasoning is applied when the four fingers are placed on two pairs of parallel edges, obtaining a total of four extremes. Then, the nominal ranges of the four torques are Infinite. ◇

It can be easily determined which nominal range is Infinite and which is Limited knowing the relative directions of the applied forces: the two applied forces (let's say $\boldsymbol{f}_h$ and $\boldsymbol{f}_i$) that lies between the negated other two (let's say $-\boldsymbol{f}_j$ and $-\boldsymbol{f}_k$), have the nominal ranges R_{fc_h} and R_{fc_i} Infinite and R_{fc_j} and Rfc_k are Limited (the coefficients of equation (11) determined by $\{\boldsymbol{f}_h, \boldsymbol{f}_j, \boldsymbol{f}_k\}$ and $\{\boldsymbol{f}_i, \boldsymbol{f}_j, \boldsymbol{f}_k\}$ are always negative). Figure 2.a shows an example.

Lemma 1 If one of the two Infinite ranges tends to $\pm\infty$ the other one tends to $\mp\infty$. ◇

Proof: Consider that R_{fc_h} and R_{fc_i} are Infinite and

R_{fc_j} and R_{fc_k} are Limited. It is not known a priori if $R_{fc_j} = [\tau_{j_1}, \tau_{j_2}]$ or $R_{fc_j} = [\tau_{j_2}, \tau_{j_1}]$, then the two cases are considered.

If $R_{fc_j} = [\tau_{j_1}, \tau_{j_2}]$ then $\tau_{j_1} \leq \tau_j \leq \tau_{j_2}$. Substituting τ_{j_1} and τ_{j_2} by their expressions derived from equation (11), we obtain

$$\beta_{h,j_1}\tau_h + \beta_{k,j_1}\tau_k \leq \tau_j \leq \beta_{i,j_2}\tau_i + \beta_{k,j_2}\tau_k \qquad (14)$$

If τ_h and τ_i are solved from equation (14), then

$$\tau_h \leq \frac{1}{\beta_{h,j_1}}(\tau_j - \beta_{k,j_1}\tau_k) \qquad (15)$$

$$\tau_i \geq \frac{1}{\beta_{i,j_2}}(\tau_j - \beta_{k,j_2}\tau_k) \qquad (16)$$

Therefore, τ_h has an upper bound while τ_i has a bottom bound.

If $R_{fc_j} = [\tau_{j_2}, \tau_{j_1}]$ then, with the same reasoning, the equations (15) and (16) are obtained again but their inequalities are changed. Then, τ_h has a bottom bound while τ_i has an upper bound. Same reasoning can be made starting from R_{fc_k}.

As a result, the two Infinite nominal ranges always tend to infinite with different signs. ◇

Note that there is no relation between τ_h and τ_i. Then, the values of τ_h and τ_i do not influence the nominal range of each other.

From Lemma 2, the following necessary and sufficient condition for the existence of a FC grasp can be enunciated.

Necessary and sufficient condition: Four frictionless contacts allow a FC grasp if and only if

$$sign(\delta) \neq sign(\epsilon) \qquad (17)$$

with

$$\delta = \beta_{j,h_1}\tau_j + \beta_{k,h_1}\tau_k - \tau_h \qquad (18)$$

$$\epsilon = \beta_{j,i_2}\tau_j + \beta_{k,i_2}\tau_k - \tau_i \qquad (19)$$

where τ_j and τ_k have Limited nominal ranges and τ_h and τ_i have Infinite nominal ranges. ◇

Geometrically, equations (18) and (19) represent two planes in two different 3-dimensional subspaces when δ and ϵ are zero. These subspaces are defined by $\{\tau_j, \tau_k, \tau_h\}$ and $\{\tau_j, \tau_k, \tau_i\}$, where τ_j and τ_k have Limited nominal ranges and τ_h and τ_i have Infinite nominal ranges. Let R_γ, $\gamma = \{h, i, j, k\}$, be the range of τ_γ that is physically possible due to the edge length. Then, the following polytopes are defined (see Fig. 3):
$S_h^+ = \{\{\tau_h, \tau_i, \tau_j, \tau_k\} | \tau_h \in R_h, \tau_j \in R_j, \tau_k \in R_k, \delta \geq 0\}$.
$S_h^- = \{\{\tau_h, \tau_i, \tau_j, \tau_k\} | \tau_h \in R_h, \tau_j \in R_j, \tau_k \in R_k, \delta \leq 0\}$.
$S_i^+ = \{\{\tau_h, \tau_i, \tau_j, \tau_k\} | \tau_i \in R_i, \tau_j \in R_j, \tau_k \in R_k, \epsilon \geq 0\}$.
$S_i^- = \{\{\tau_h, \tau_i, \tau_j, \tau_k\} | \tau_i \in R_i, \tau_j \in R_j, \tau_k \in R_k, \epsilon \leq 0\}$.

The geometrical interpretation of equation (17) is

$$(S_h^+ \cap S_i^-) \cup (S_h^- \cap S_i^+) \neq \emptyset \qquad (20)$$

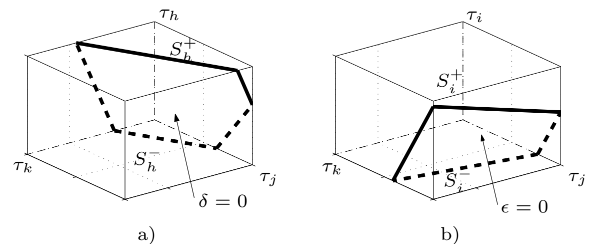

Figure 3: *Polyhedrons resulting from the projection of: a) S_h^+ and S_h^- on the subspace $\{\tau_j, \tau_k, \tau_h\}$; b) S_i^+ and S_i^- on the subspace $\{\tau_j, \tau_k, \tau_i\}$.*

3.3 Two unknown friction contacts

The wrenches produced by two forces applied on two friction contact points can be expressed as a linear combination of four primitive wrenches (Subsection 2.1.). Considering these four primitive wrenches, the necessary and sufficient condition that ensures a FC grasp for four frictionless contacts is also applicable for two friction contacts. Besides, the relation of dependency between the two primitive wrenches associated to one force has also to be accomplished in order to obtain a real FC grasp. This dependency can be considered introducing equations (6) and (7) in the necessary and sufficient condition for four frictionless contacts given by equation (17).

Let $\boldsymbol{\omega}_{i,l}$ and $\boldsymbol{\omega}_{i,r}$ be the pair of primitive wrenches at contact point i, and let $\boldsymbol{\omega}_{j,l}$ and $\boldsymbol{\omega}_{j,r}$ be the pair of primitive wrenches at contact point j. Since $\varphi \leq \pi$ ($\varphi = \arctan \mu$) the torque components of the primitive wrenches on the same point have different type of nominal ranges (see Fig. 2.b). Then, the expressions for δ and ϵ when two friction contacts are considered are:

$$\delta = \beta_{i_r,i_l}(\tau_{i,n}+\tau_{i,t}) + \beta_{j_l,i_l}(\tau_{j,n}-\tau_{j,t}) - (\tau_{i,n}-\tau_{i,t}) \qquad (21)$$

$$\epsilon = \beta_{i_r,j_r}(\tau_{i,n}+\tau_{i,t}) + \beta_{j_l,j_r}(\tau_{j,n}-\tau_{j,t}) - (\tau_{j,n}+\tau_{j,t}) \qquad (22)$$

where $\tau_{i,n}$ and $\tau_{j,n}$ are the torques produced by the normal forces, and $\tau_{i,t}$ and $\tau_{j,t}$ are the torques produced by the tangent forces.

3.4 Maximum independent regions

The independent regions are defined as the segments on the object boundary such that a finger in each segment ensures a FC grasp [9]. The typical maximization criterion used in the literature, as well as here, is the maximization of the shortest segment, increasing the robustness in front of the finger positioning errors.

The algorithm used to find the independent regions is described considering frictionless contacts (friction contacts can be considered using equations (21) and (22)). Let $[\tau_p^-, \tau_p^+]$ be the independent region on the object edge p, with $p = 1, ..., 4$. The independent regions define a 4-dimensional parallelepiped (4D-parallelepiped) and it can be determined using the following proposition:

Proposition 2 Let v, $v = 1, ..., 16$, be a vertex of the 4D-parallelepiped and let δ_v and ϵ_v be the solutions of equations (18) and (19) for v. A combination of contact points inside the 4D-parallelepiped produces a FC grasp if the following condition is true $\forall v$:

$$(\delta_v \geq 0 \; and \; \epsilon_v \leq 0) \quad or \quad (\delta_v \leq 0 \; and \; \epsilon_v \geq 0) \quad (23)$$

$\diamond$

Proof: Consider the geometrical interpretation of the necessary and sufficient condition expressed in equation (20). By construction $S_h^+ \cap S_i^-$ and $S_h^- \cap S_i^+$ are convex sets, but the set $(S_h^+ \cap S_i^-) \cup (S_h^- \cap S_i^+)$ is concave. Since the 4D-parallelepiped is convex, we can guarantee that all its points satisfy the necessary and sufficient condition if all its vertices belong to a convex solution set, i.e., all the vertices belong to $S_h^+ \cap S_i^-$ or to $S_h^- \cap S_i^+$, which can not be true when the vertices of the 4D-parallelepiped belong to $(S_h^+ \cap S_i^-) \cup (S_h^- \cap S_i^+)$. Thus, if the 4D-parallelepiped is inside $S_h^+ \cap S_i^-$ then $\forall v \; \delta_v \geq 0$ and $\epsilon_v \leq 0$, and if the 4D-parallelepiped is inside $S_h^- \cap S_i^+$ then $\forall v \; \delta_v \leq 0$ and $\epsilon_v \geq 0$. $\diamond$

The algorithm developed to determine the maximal independent regions is based on linear programming. Proposition 2 defines a set of logical relations between the variables (the vertices of the 4D-parallelepiped) and they can be introduced in the Simplex algorithm if equation (23) is expressed as:

$$\text{If } \exists \delta_v \geq 0 \text{ Then } \forall \delta_v \geq 0 \text{ and } \forall \epsilon_v \leq 0 \quad (24)$$
$$\text{If } \exists \delta_v \leq 0 \text{ Then } \forall \delta_v \leq 0 \text{ and } \forall \epsilon_v \geq 0 \quad (25)$$
$$\text{If } \exists \epsilon_v \geq 0 \text{ Then } \forall \epsilon_v \geq 0 \text{ and } \forall \delta_v \leq 0 \quad (26)$$
$$\text{If } \exists \epsilon_v \leq 0 \text{ Then } \forall \epsilon_v \leq 0 \text{ and } \forall \delta_v \geq 0 \quad (27)$$

This kind of relations are denominated If-Then constraints in linear programming [12]. Let F and G be two generic functions related with a If-Then Constraint (e.g., If $F \geq 0$ Then $G \geq 0$), this constraint is expressed in the Simplex algorithm as:

$$F \leq M(1-y) \quad (28)$$
$$-G \leq My \quad (29)$$

where y is a binary variable and M is a sufficient large value.

Let L be the minimum length of an independent region on a edge. Then, the algorithm used to determine the maximum independent regions has the following form, for $p = 1, ..., 4$:

Max L

subject to

$L \leq \tau_p^+ - \tau_p^-$

$\tau_p^- \leq \tau_p^+$

$\tau_p^+ \leq \tau_{max_p}$

$\tau_{min_p} \leq \tau_p^-$

Eq. (24), (25) (26) and (27)

4 Redundant grasps

Grasps formed by more than four frictionless contacts and more than two friction contacts are redundant grasps since they can apply forces in more than four directions. The redundant grasps can be decomposed into a set of non-redundant grasps considering the combinations of four wrenches that ensure a FC grasp. In order to obtain all the non-redundant grasps of a given redundant grasp, it is initially considered that all the primitive wrenches are independent (even if friction contacts are considered). For each combination of two wrenches the following procedure is applied (the obtention of these combinations has a computational cost $O(N^2)$):

1. Consider that the nominal ranges of the torque components of the two wrenches are Limited and find from the remaining wrenches those whose torque components have Infinite nominal ranges (it can be done knowing the relative directions of the applied forces, as it was shown in Fig. 2). The computational cost of this step is $O(N)$.
2. Depending of the number of remaining wrenches whose torque components have an Infinite nominal ranges, do:
   - 2.1. If at most one wrench is found, then this combination is rejected since at least four wrenches are necessary to form a FC grasp.
   - 2.2. If at least two wrenches are found, then each combination of the two initial wrenches with two wrenches whose torque components have an Infinite nominal range forms a non-redundant FC grasp.
3. For each non-redundant grasp, determine the necessary and sufficient condition by computing the coefficients of equations (18) and (19).
4. If friction contacts are considered, use equations (6) and (7) in order to express the necessary and sufficient condition as a function of the torques produced by forces orthogonal to the edge (then, the dependency on the pairs of primitive wrenches generated by the same force is considered).

As a result, the non-redundant grasps are obtained with a computational cost of $O(N^3)$.

The linear programming algorithm used to obtain the maximum independent regions described in Subsection 3.4 is applied for each of the non-redundant grasps, choosing as the best set of independent regions the set with the largest minimum region length.

5 Examples

A numerical example of the proposed methodology is presented in this section, determining the independent regions for three friction contacts. The friction coefficient is $\mu = 0.3$ and the three edges where the three fingers will contact are known. Thus, for each finger i, we know the contact edge i and (see the object in Fig. 4.a):

i	θ_i	$\theta_{i,r}$	$\theta_{i,l}$	$\tau_{i,n_{min}}$	$\tau_{i,n_{max}}$	$\tau_{i,t}$
1	1.570	1.279	1.862	-0.603	2.196	0.331
2	4.112	3.820	4.403	-2.463	-0.160	0.083
3	4.957	4.665	5.248	0.032	1.204	0.278

where θ_i is the direction of the force orthogonal to the edge, $\theta_{i,r}$ and $\theta_{i,l}$ are the directions of the primitive forces, $\tau_{i,n_{min}}$ and $\tau_{i,n_{max}}$ are the minimum and maximum torques, respectively, produced by a force orthogonal to the edge, and $\tau_{i,t}$ is the torque produced by the force tangent to the edge.

All the non-redundant grasps are produced by the following combinations of primitive wrenches:

1. $\{\omega_{1,r}, \omega_{2,r}, \omega_{3,r}, \omega_{3,l}\}$	7. $\{\omega_{1,r}, \omega_{1,l}, \omega_{3,r}, \omega_{3,l}\}$
2. $\{\omega_{1,r}, \omega_{2,l}, \omega_{3,r}, \omega_{3,l}\}$	8. $\{\omega_{1,l}, \omega_{2,r}, \omega_{2,l}, \omega_{3,l}\}$
3. $\{\omega_{1,r}, \omega_{1,l}, \omega_{2,r}, \omega_{3,r}\}$	9. $\{\omega_{1,l}, \omega_{2,r}, \omega_{3,r}, \omega_{3,l}\}$
4. $\{\omega_{1,r}, \omega_{1,l}, \omega_{2,l}, \omega_{3,r}\}$	10. $\{\omega_{1,l}, \omega_{2,l}, \omega_{3,r}, \omega_{3,l}\}$
5. $\{\omega_{1,r}, \omega_{2,r}, \omega_{2,l}, \omega_{3,r}\}$	11. $\{\omega_{1,r}, \omega_{1,l}, \omega_{2,r}, \omega_{3,l}\}$
6. $\{\omega_{1,r}, \omega_{2,r}, \omega_{2,l}, \omega_{3,l}\}$	12. $\{\omega_{1,r}, \omega_{1,l}, \omega_{2,l}, \omega_{3,l}\}$

Considering combination 4, $\tau_{1,r}$ and $\tau_{3,r}$ have Limited nominal ranges and $\tau_{1,l}$ and $\tau_{2,l}$ have Infinite nominal ranges. The values of δ and ϵ are given by:

$$\delta = -1.3670\tau_{1,r} - \tau_{1,l} - 2.2696\tau_{3,r} \quad (30)$$

$$\epsilon = -1.0682\tau_{1,r} - \tau_{2,l} - 0.0705\tau_{3,r} \quad (31)$$

and using equations (6) and (7) to express δ and ϵ as a function of $\tau_{i,n}$, we obtain:

$$\delta = -2.3670\tau_{1,n} - 2.2696\tau_{3,n} - 0.7536 \quad (32)$$

$$\epsilon = -1.0682\tau_{1,n} - \tau_{2,n} - 0.0705\tau_{3,n} - 0.2902 \quad (33)$$

Equations (32) and (33) establish the necessary and sufficient condition for combination 4, and it is satisfied if the signs of δ and ϵ are different.

With the same reasoning, the necessary and sufficient conditions for the other combinations are found. The result of the Simplex algorithm considering the necessary and sufficient condition for each combination and the minimum and maximum values of $\tau_{i,n}$ are:

Comb.	L	$[\tau_1^-, \tau_1^+]$	$[\tau_2^-, \tau_2^+]$	$[\tau_3^-, \tau_3^+]$
4.	1.172	$[-0.349, 0.856]$	$[-2.463, -1.290]$	$[0.032, 1.204]$
5.	0.978	$[0.059, 1.038]$	$[-2.463, -1.484]$	$[0.226, 1.204]$
1.	0.636	$[-0.198, 0.437]$	$[-2.463, -1.826]$	$[0.568, 1.204]$
12.	0.619	$[0.925, 1.544]$	$[-2.463, -1.843]$	$[0.585, 1.204]$
10.	0.565	$[-0.290, 0.274]$	$[-2.463, -1.897]$	$[0.639, 1.204]$
6.	0.486	$[1.189, 1.675]$	$[-2.463, -1.976]$	$[0.718, 1.204]$
9.	0.405	$[-0.457, -0.051]$	$[-2.463, -2.057]$	$[0.799, 1.204]$
7.	0.338	$[-0.349, -0.010]$	$[-2.463, -0.160]$	$[0.032, 0.370]$
11.	0.322	$[0.349, 0.671]$	$[-2.463, -2.140]$	$[0.882, 1.204]$
3.	0.293	$[-0.603, -0.310]$	$[-2.463, -2.169]$	$[0.297, 0.590]$
8.	0.181	$[0.217, 0.398]$	$[-2.463, -2.281]$	$[1.023, 1.204]$
2.	0.097	$[1.961, 2.058]$	$[-2.463, -2.365]$	$[1.107, 1.204]$

L being the length of the smallest independent region, and $[\tau_1^-, \tau_1^+]$, $[\tau_2^-, \tau_2^+]$ and $[\tau_3^-, \tau_3^+]$ the independent regions on each edge. Combination 4 produces the best result, which in this case makes the smallest region on one edge to be equal to the edge itself. Figure 4 shows the independent regions on the object boundary and the parallelepiped that they form in the space defined by $\{\tau_{1,n}, \tau_{2,n}, \tau_{3,n}\}$.

6 Conclusions

The paper provides a new approach to determine the independent regions on a given set of edges of an object that allow a FC grasp for N fingers. All the possible FC grasps are considered in this approach, although it is not necessary to construct the entire

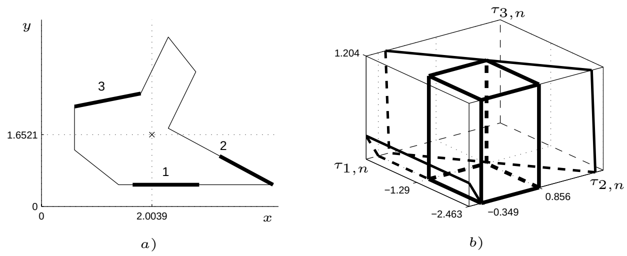

Figure 4: *Results of combination 4: a) Independent regions on the object boundary; b) Parallelepiped in the space defined by* $\{\tau_{1,n}, \tau_{2,n}, \tau_{3,n}\}$.

N-dimensional set of the FC grasps. The grasp is decomposed into several non-redundant grasps, and each one establishes an individual necessary and sufficient condition for force-closure. If at least one of them is satisfied, then a global FC grasp is obtained. These conditions are included in an algorithm based on linear programming that is used to compute the independent regions.

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